Transformations between Signed and Classical Clause Logic*

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Abstract. In the last years two automated reasoning techniques for clause normal form arose in which the use of labels are prominently featured: *signed logic* and *annotated logic programming*, which can be embedded into the first. The underlying basic idea is to generalize the classical notion of a literal by adorning an atomic formula with a sign or label which in general consists of a possibly ordered set of truth values. In this paper we relate signed logic and classical logic more closely than before by defining two new transformations between them. As a byproduct we obtain a number of new complexity results and proof procedures for signed logics.

1 Introduction

In the last years two automated reasoning techniques for clause normal form arose in which the use of labels are prominently featured: from generic treatments of many-valued logic, so-called *signed logic* emerged (see, for example, [7, 8, 4, 5, 15, 16]) while *annotated logic programming* (see, for example, [13, 9, 10]) was motivated by attempts to deal with inconsistency in deductive databases. Both approaches are closely connected to each other [14, 11] and to constraint logic programming [12]. In fact, annotated logic can be embedded into signed logic [14].

In any case the underlying basic idea is to generalize the classical notion of a literal by adorning an atomic formula with a sign or label which in general consists of a finite set of (truth) values. Whenever the values appearing in the signs are partially ordered, polarities can be assigned to signed literals in a natural way which gives rise to generalized notions of a Horn set. It turns out that many problems can be represented more succinctly using formulae over signed literals whose proof procedures and complexity are often (but not always) similar as in classical logic.

In the present paper we relate signed logic and classical logic more closely than it has been done before. This is done by defining two new transformations between them. After formal definition of some basic notions in the next section

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we start in Section 3 with transforming arbitrary classical formulae in conjunctive normal form (CNF) into signed CNF formulae *with at most two literals per clause*. This provides an alternative proof of NP-hardness of signed 2-SAT (first proved by [15]) and creates the possibility to compare classical and signed deduction procedures experimentally. In Section 4.1 we take the reverse direction and reduce signed Horn formulae based on certain partial orders to classical logic. In the case of lattice orders this yields the new result that generalized Horn problems turn out to have still polynomial complexity with respect to formula size and number of truth values (Section 4.2). We can also extract an efficient decision procedure based on generalized unit resolution (Section 4.4). A major advantage of our reduction to classical logic is that it scales up: we demonstrate this by sketching generalizations to infinite orders in Section 4.5 and to partial orders that are not lattices in Section 4.6.

2 Basic Definitions

2.1 Syntax of Signed Logic

Definition 1. *A* truth value set *N is a finite set $\{i_1, i_2, \ldots, i_n\}$, where $n \in \mathbb{N}$. The cardinality of N is denoted by $|N|$.*

Definition 2. *Let Σ be a* propositional signature*, that is, a denumerable set of propositional variables. We define the* set of atomic signed formulae *(or signed atoms for short) as the following set:*

$$\{S : p_i \mid S \subseteq N, p_i \in \Sigma\}.$$

Definition 3. *Given a signed atom $S : p$, then S is said to be its sign. Let $\geq$ be a partial order on the truth value set N, let $\uparrow i$ denote the set $\{j \in N \mid j \geq i\}$ and let $\downarrow i$ denote the set $\{j \in N \mid j \leq i\}$. If a sign S is equal to either $\uparrow i$ or $\downarrow i$, for some $i \in N$, then it is called a* regular sign. *A signed atom with a regular sign is called a* regular atom.

Definition 4. *A* signed clause *C is an expression of the form*

$$S_1 : p_1, \ldots, S_k : p_k \;\rightarrow\; S'_1 : q_1, \ldots, S'_l : q_l$$

where $S_1 : p_1, \ldots, S_k : p_k$ and $S'_1 : q_1, \ldots, S'_l : q_l$ are signed atoms and $k, l \geq 0$. The signed atoms $S_1 : p_1, \ldots, S_k : p_k$ are said to occur in C with negative polarity, *and the signed atoms $S'_1 : q_1, \ldots, S'_l : q_l$ are said to occur with* positive polarity. *The expression on the left of $\rightarrow$ is called the* body *of the clause and the expression on the right is called the* head. *A* signed formula *in conjunctive normal form (CNF) is a finite set of signed clauses.*

A signed clause is called regular *if $(N, \geq)$ is a lattice and it only contains regular atoms with signs of the form $\uparrow i$.*[1] *A signed CNF formula is called* regular *if it only contains regular clauses. A regular clause containing at most one atom with positive polarity is a* regular Horn clause. *A regular CNF formula consisting solely of regular Horn clauses is a* regular Horn formula.

[1] Regular clauses could also be defined containing only signs of the form $\downarrow i$ instead of signs of the form $\uparrow i$. The results of this paper are also valid for regular clauses defined that way.

Note that in clauses both $k = 0$ and $l = 0$ is allowed; thus, $p, q \rightarrow \langle\rangle$ and $\langle\rangle \rightarrow p, q$ are signed clauses, and we represent them by $p, q \rightarrow$ and $\rightarrow p, q$. When $k = 0$ and $l = 0$ we have the signed empty clause, denoted by $\Box$.

Definition 5. *The* length *of a* signed atom $S : p$*, denoted by* $|S : p|$*, is* $|S| + 1$*, where* $|S|$ *denotes the cardinality of* S*. The* length *of a* signed clause C*, denoted by* $|C|$*, is the sum of the lengths of the signed atoms occurring in* C*. The* length *of a* signed CNF formula Γ*, denoted by* $|\Gamma|$*, is the sum of the lengths of the clauses of* Γ*.*

2.2 Semantics of Signed Logic

Definition 6. *An* interpretation *is a mapping that assigns to every propositional variable of* Σ *a truth value of* N*. An interpretation* I satisfies *a* signed atom $S : p$*, in symbols* $I \models S : p$*, iff* $I(p) \in S$*. An interpretation* I satisfies *a* signed clause $C = S_1 : p_1, \ldots, S_k : p_k \rightarrow S'_1 : q_1, \ldots, S'_l : q_l$*, in symbols* $I \models C$*, iff the following condition holds: If* I *satisfies all the signed atoms* $S_1 : p_1, \ldots, S_k : p_k$ *then* I *satisfies at least one of the signed atoms* $S'_1 : q_1, \ldots, S'_l : q_l$*. A signed CNF formula* Γ *is* satisfiable *iff there exists an interpretation* I *that satisfies all the signed clauses in* Γ*. We say then that* I *is a model of* Γ *and we write* $I \models \Gamma$*. A signed CNF formula that is not satisfiable is* unsatisfiable*. The signed empty clause is always unsatisfiable and the signed empty CNF formula is always satisfiable.*

Note that I satisfies $\rightarrow S'_1 : q_1, \ldots, S'_l : q_l$ iff it satisfies at least one of the signed atoms $S'_1 : q_1, \ldots, S'_l : q_l$ and I satisfies $S_1 : p_1, \ldots, S_k : p_k \rightarrow$ iff it does *not* satisfy all the signed atoms $S_1 : p_1, \ldots, S_k : p_k$.

Observe that if we take $N = \{true, false\}$, assuming $true > false$, and consider only regular atoms of the form $\uparrow true : p$, then we obtain the logic of classical conjunctive normal forms: $\uparrow true : p$ is equivalent to the classical atom p if it occurs with positive polarity, and to the negated classical atom $\neg p$ if it occurs with negative polarity. So, the classical clause $p_1, \ldots, p_k \rightarrow q_1, \ldots, q_l$ is equivalent to the regular clause $\uparrow true : p_1, \ldots, \uparrow true : p_k \rightarrow \uparrow true : q_1, \ldots, \uparrow true : q_l$. In the following, when we refer to classical clauses we use the former notation.

In classical propositional logic, clauses are also defined as a finite disjunction of literals (i.e. signed atoms or negated signed atoms). It is easy to see from the previous definitions that $p_1, \ldots, p_k \rightarrow q_1, \ldots, q_l$ is logically equivalent to $\neg p_1 \vee \cdots \vee \neg p_k \vee q_1 \vee \cdots \vee q_l$. So, classical atoms occurring with negative polarity are implicitly negated. In our definition of signed clauses, signed atoms occurring with negative polarity are also implicitly negated in the sense that a signed atom $S : p$ with negative polarity is satisfied by an interpretation I iff $I \not\models S : p$ When we focus on the subclass of regular clauses we take the same approach: We consider regular atoms with a sign of the form $\uparrow i$ and we have that an occurrence of $\uparrow i : p$ with negative polarity is satisfied by an interpretation I iff $I \not\models \uparrow i : p$. The same holds when regular clauses contain only atoms of the form $\downarrow i : p$. Nevertheless, we do not define regular clauses as a disjunction of regular atoms with arbitrary regular signs since, as we assume a partial order $\geq$ on N, an occurrence of $\uparrow i : p$ with negative polarity is not, in general, logically equivalent to $\downarrow j : p$ for some $j \in N$. If we assume a total order, then it holds and it is usual to represent regular clauses as a disjunction of regular atoms.

2.3 Satisfiability Problems

The propositional satisfiability (SAT) problem is the problem of determining whether a classical CNF formula is satisfiable, and is known for being the original NP-complete problem [1]. However, there exist linear-time algorithms for solving the SAT problem when we consider Horn formulae (Horn SAT) [3] or CNF formulae with only two literals per clause (2-SAT) [6]. When the CNF formula has three literals per clause (3-SAT), it is again an NP-complete problem.

In the last years, some results about the complexity of the propositional satisfiability problem for signed CNF formulae (signed SAT) have been published. The signed SAT problem[2] and the signed 2-SAT problem [15] are NP-complete, but when signs are singletons the signed 2-SAT problem (monosigned 2-SAT) [15] is polynomially solvable. Concerning the regular case, it is known that the regular Horn SAT problem [8, 4] and the regular 2-SAT problem [15] are both polynomially solvable when the partial order defined over the set of truth values is total.

3 Transforming Classical SAT into Signed 2-SAT

3.1 The Transformation

In this section, we define a mapping $'$ transforming classical CNF formulae into signed (non-regular) 2-CNF formulae; and we prove that it is a poly-time reduction.

The mapping $'$ is defined as follows: Let Γ be a classical CNF formula with clauses $C_1, \ldots, C_r$ $(r \geq 1)$ over a signature Σ. Assume that $p_1, \ldots, p_s$ $(s \geq 1)$ are the propositional variables occurring in Γ; thus, the clauses in Γ are of the form[3]

$$C_m = p_{i_{m,1}}, \ldots, p_{i_{m,k_m}} \rightarrow p_{j_{m,1}}, \ldots, p_{j_{m,l_m}} \quad .$$

We associate with Γ a signed 2-CNF formula Γ' over the truth value set $N = \{p_1^-, \ldots, p_s^-, p_1^+, \ldots, p_s^+\}$ and signature $\Sigma' = \{p'_1, \ldots, p'_r\}$, i.e., the truth values are the classical atoms annotated with the two possible polarities $-$ and $+$, and for each clause C_m in Γ there is a propositional variable p'_m in Σ'. The idea is that p'_m has the truth value p_i^+ resp. p_i^- in a (non-classical) interpretation I' if the classical atom p_i is the one that makes the clause C_m true in the corresponding classical interpretation I; that is, $I'(p'_m) = p_i^-$ if p_i is false in I and occurs with negative polarity in C_m, and $I'(p'_m) = p_i^+$ if p_i is true in I and occurs with positive polarity in C_m. An atom can only have a single truth value whereas a clause C_m can be "made true" by more than one of its literals, in which case an arbitrary one may be chosen to be the truth value of p'_m.

For each clause $C_m = p_{i_{m,1}}, \ldots, p_{i_{m,k_m}} \rightarrow p_{j_{m,1}}, \ldots, p_{j_{m,l_m}}$ in Γ there is a singleton clause

$$(C'_m) \qquad \rightarrow \{p^-_{i_{m,1}}, \ldots, p^-_{i_{m,k_m}}, p^+_{j_{m,1}}, \ldots, p^+_{j_{m,l_m}}\} : p'_m$$

in Γ'.

[2] It is straightforward to see the NP-hardness of the signed SAT problem by proving that the classical SAT problem is polynomially reducible to it.

[3] Note, that the atoms in Γ are actually signed atoms of the form $\uparrow true : p$; but, as said in Section 2, the signs are not shown in representations of classical CNF formulae.

The signed atom in C'_m represents the fact that C_m (the m-th clause of Γ) is made true. Thus Γ' represents only satisfying truth assignments of Γ.

This is, of course, not enough. We must ensure that Γ' in fact represents solely such truth assignments for atoms in Γ which are consistent or, in usual terminology, which are well-defined interpretations. For this purpose, Γ' contains the following additional clauses for all (classical) clauses C_m and C_n in Γ resp. for all propositional variables p'_m and p'_n in Σ' ($1 \leq m, n \leq r$) and for all atoms resp. truth values p_i ($1 \leq i \leq s$):

$$(D'_{mni}) \qquad \{p_i^+\} : p'_m \rightarrow (\{p^-_{i_{n,1}}, \ldots, p^-_{i_{n,k_n}}, p^+_{j_{1,n}}, \ldots, p^+_{j_{n,l_n}}\} \setminus \{p_i^-\}) : p'_n$$

The signed clauses D'_{mni} express that if an atom is used with positive polarity to "make true" some clause C_m of Γ, then it cannot be used with negative polarity to "make true" any other clause of Γ.

The clause D'_{mni} may be omitted from Γ' if p_i does not occur with positive polarity in C_m or does not occur with negative polarity in C_n,

Instead of the clauses D'_{mni} the clauses

$$(E'_{mni}) \qquad \{p_i^-\} : p'_m \rightarrow (\{p^-_{i_{n,1}}, \ldots, p^-_{i_{n,k_n}}, p^+_{j_{1,n}}, \ldots, p^+_{j_{n,l_n}}\} \setminus \{p_i^+\}) : p'_n$$

can be used. The proof of Theorem 2 shows that it is indeed sufficient to either use only the clauses D'_{mni} or only the clauses E'_{mni}.

Example 1. Consider the classical CNF formula Γ consisting of the clauses

$$\begin{array}{ll} (C_1) & p \rightarrow q \\ (C_2) & q \rightarrow p \\ (C_3) & \rightarrow p, q \end{array}$$

The only model of Γ is the interpretation I with $I(p) = I(q) = \mathit{true}$. The result of transforming Γ is a signed 2-SAT formula Γ' over the signature $\Sigma' = \{p'_1, p'_2, p'_3\}$ and using the truth value set $N = \{p^-, q^-, p^+, q^+\}$; Γ' consists of the clauses

$$\begin{array}{ll} (C'_1) & \rightarrow \{p^-, q^+\} : p'_1 \\ (C'_2) & \rightarrow \{q^-, p^+\} : p'_2 \\ (C'_3) & \rightarrow \{p^+, q^+\} : p'_3 \\ (D'_{211}) & \{p^+\} : p'_2 \rightarrow \{q^+\} : p'_1 \\ (D'_{311}) & \{p^+\} : p'_3 \rightarrow \{q^+\} : p'_1 \\ (D'_{122}) & \{q^+\} : p'_1 \rightarrow \{p^+\} : p'_2 \\ (D'_{322}) & \{q^+\} : p'_3 \rightarrow \{p^+\} : p'_2 \end{array}$$

In (non-classical) interpretations I' satisfying Γ', the truth value of p'_1 is q^+, and the truth value of p'_2 is p^+. The truth value of p_3 can be either p^+ or q^+, according to the fact that both atoms in the clause C_3 are satisfied by the classical interpretation I.

3.2 Results

The size of Γ' is easily seen to be

$$\sum_m \underbrace{(k_m + l_m + 1)}_{=|C'_m|} + \sum_{m,n,i} \underbrace{(k_n + l_n + 2)}_{=|D'_{mni}|} \leq |\Gamma| + r + 2r^2 s$$

where r is the number of clauses in Γ, and s is the number of different atoms occurring in Γ. As $r, s < |\Gamma|$, this places $|\Gamma'|$ in $\mathcal{O}(|\Gamma|^3)$. Obviously, Γ' can be constructed in time which is linear in its own size and, thus, the time complexity of its construction is in $\mathcal{O}(|\Gamma|^3)$.

Theorem 1. *The transformation $'$ is computable in cubic time.*

Signed (non-regular) 2-SAT was proven to be NP-hard in [15] by providing a poly-time reduction from 3-colourability of graphs to signed 2-CNF. As classical SAT is NP-complete, the same result follows as a corollary from Theorem 1.

Corollary 1. *Signed 2-SAT is NP-complete.*

An additional benefit of the transformation $'$ is that it makes it possible to compare classical decision procedures with specific procedures for signed CNF.

3.3 Correctness

The following theorem states the correctness of the transformation $'$:

Theorem 2. *A classical CNF formula Γ is satisfiable if and only if Γ' is satisfiable.*

Proof. 1. *Only-if-part:* Assume that the classical interpretation I satisfies Γ. Define the interpretation I' as follows. In each clause $C_m \in \Gamma$ there has to be an atom p such that (1) $I(p) = true$ and p occurs positively in C_m or (2) $I(p) = false$ and p occurs negatively in C_m, because otherwise C_m were not satisfied by I. If there is more than one such atom p in C_m, then choose an arbitrary one. If (1) holds for p, then define $I'(p'_m) = p^+$, otherwise (i.e., if (2) holds for p) define $I'(p'_m) = p^-$.

a. $I' \models C'_m$: If $I'(p'_m) = p^+$ (resp. $I'(p'_m) = p^-$), then p occurs positively (negatively) in C_m and, thus, p^+ (resp. p^-) is an element of the truth value sign attached to p'_m in C'_m.

b. $I' \models D'_{mni}$: Let $1 \leq m \leq r$ and $1 \leq i \leq s$ be arbitrary. If $I'(p'_m) \neq p_i^+$, then D'_{mni} is trivially satisfied for all n. Otherwise, $I(p'_m) = p_i^+$ and we must show that $I'(p'_n) \in \{p^-_{i_{n,1}}, \ldots, p^-_{i_{n,k_n}}, p^+_{j_{1,n}}, \ldots, p^+_{j_{n,l_n}}\} \setminus \{p_i^-\}$ for all n. We know that $I'(p'_n)$ is in $\{p^-_{i_{n,1}}, \ldots, p^-_{i_{n,k_n}}, p^+_{j_{1,n}}, \ldots, p^+_{j_{n,l_n}}\}$, as $I' \models C_n$ for all n. It remains to be shown that $I'(p'_n) \neq p_i^-$; to produce a contradiction, assume $I'(p'_n) = p_i^-$ for some n. By definition of I', this implies $I(p_i) = false$. On the other hand, we have $I'(p'_m) = p_i^+$ implying $I(p_i) = true$, which is a contradiction.

2. *If-part:* Assume that the interpretation I' satisfies Γ'. Define the classical interpretation I for all atoms $p \in \Sigma$ as follows: if there is an atom p'_m $(1 \leq m \leq r)$ such that $I'(p'_m) = p^+$, then let $I(p) = true$; otherwise let $I(p) = false$. It remains to be shown that I satisfies all clauses C_m in Γ.

a. If $I'(p'_m) = p^+$, then (1) p occurs with positive polarity in C_m (by definition of $'$ and because $I' \models C'_m$), and (2) $I(p) = true$ (by definition of I). Thus, $I \models C_m$.

b. Otherwise, if $I'(p'_m) = p^-$, then (1) p occurs with negative polarity in C_m (for the same reasons as in (a) above), but now (2) $I(p) = false$ is harder to show. Assume the contrary, i.e., $I(p) = true$; that is only possible if there is an atom p'_n such that $I'(p'_n) = p^+$. But then the clause D'_{nmi} (where $p = p_i$) is not satisfied by I', which contradicts the assumption that $I' \models \Gamma'$. Thus, (2) $I(p) = false$ holds; and (1) and (2) imply that $I \models C_m$. $\square$

4 Transforming Regular Horn SAT into Classical Horn SAT

4.1 The Transformation

In this section, we define a mapping * transforming lattice-based regular Horn formulae into classical Horn formulae; and we prove that it is linear in the size of the signature and quadratic in the size of the truth-value lattice.

We assume in the following that the formula to be transformed does not contain a signed atom of the form $\uparrow - : p$ where $-$ is the bottom element of the truth value lattice. This is not a real restriction, as such atoms are true in all interpretations; they can be removed from a formula in linear time preserving satisfiability as follows: (1) if a clause contains a negative occurrence of $\uparrow - : p$, then remove that occurrence from the clause; (2) if a clause contains a positive occurrence of $\uparrow - : p$, then remove the whole clause from the formula.

The mapping * is defined as follows: Let Γ be a regular Horn formula over the truth-value lattice $(N, \geq)$ not containing the sign $\uparrow -$. Let $C_1, \dots, C_r$ be the clauses in Γ $(r \geq 1)$, let $p_1, \dots, p_s \in \Sigma$ be the propositional variables occurring in Γ $(s \geq 1)$.

We associate with Γ a classical Horn formula Γ^* over the signature

$$\Sigma^* = \{\uparrow i : p \mid i \in N, p \in \Sigma\} \enspace ,$$

i.e., the signed atoms—includings their signs—are used as propositional variables.

The classical formula Γ^* contains the clauses $C_m^* = C_m$ from Γ $(1 \leq m \leq r)$, which in Γ^* are regarded as classical clauses over Σ^*. In addition, Γ^* contains, for all truth values $i, j \in N$ and all propositional variables p_k occurring in Γ $(1 \leq k \leq s)$,

1. if (a) $i > j$ and (b) there is no $j' \in N$ such that $i > j' > j$, the clause

$$(D_{ijk}^*) \qquad \uparrow i : p_k \rightarrow \uparrow j : p_k \enspace ,$$

2. if neither $i \geq j$ nor $j \geq i$, the clause

$$(E_{ijk}^*) \qquad \uparrow i : p_k, \uparrow j : p_k \rightarrow \uparrow (i \sqcup j) : p_k \enspace ,$$

   where $i \sqcup j$ is the supremum of i and j in the truth value lattice.

As Γ^* contains the clauses from Γ, the classical interpretations satisfying Γ^* satisfy Γ as well. The additional clauses D_{ijk}^* and E_{ijk}^* ensure that such a classical interpretation I^* over the signature Σ^* corresponds to a well defined interpretation I over the signature Σ.

The clauses D_{ijk}^* represent the fact that, if $I \models \uparrow i : p_k$, i.e., $I(p_k) \geq i$ and $i > j$, then $I(p_k) \geq j$ and I satisfies $\uparrow j : p_k$ as well.

The clauses E_{ijk}^*, on the other hand, represent the fact that, if I satisfies both $\uparrow i : p_k$ and $\uparrow i : p_k$, i.e., $I(p_k) \geq i$ and $I(p_k) \geq j$, then $I(p_k) \geq i \sqcup j$ and, thus, $I \models \uparrow (i \sqcup j) : p_k$.

The precondition (a) $i > j$ for the inclusion of the clauses D_{ijk}^* in Γ^* is necessary for the correctness of the transformation; in case *not* $i > j$, the clauses D_{ijk}^* are (in general) not satisfied by arbitrary interpretations. Contrary to that, the precondition (b) for the inclusion of the D^*_{ijk} and the precondition for the inclusion of the clauses E^*_{ijk} is only needed to avoid redundancies.

The following lemma shows that clauses D^*_{ijk} for values of i, j violating precondition (b) are redundant. They are true in all interpretations satisfying Γ^*. Therefore, their inclusion would not impose any further restriction on the models of Γ^*.

Lemma 1. *Let Γ be a regular Horn formula over a signature Σ, let I^* be a classical interpretation satisfying Γ^*, let $p \in \Sigma$, and let j, j' be truth values in N such that $j \geq j'$ and $I^*(\uparrow j : p) = true$; then $I^*(\uparrow j' : p) = true$.*

Proof. Since $j \geq j'$ there is a sequence of truth values $j_1, \ldots, j_q \in N$, $q \geq 1$, such that $j = j_1 > \cdots > j_q = j'$ and this sequence is maximal, i.e., for $1 \leq l \leq q$, there is no $j'_l \in N$ such that $j_l > j'_l > j_{l+1}$. The proof of the lemma proceeds by induction on q.

$q = 1$: In this case $j = j'$, and the lemma is trivially true.

$q \to q + 1$: The induction hypothesis applies to the truth values j and j_q. Therefore, $I^*(\uparrow j_q : p) = true$. Because $j_q > j_{q+1} = j'$ and there is no $j'_q \in N$ such that $j_q > j'_q > j'$, the formula Γ^* contains the clause $D_{j_q j' k}$ (where $p = p_k$), which is satisfied by I^* and, since $I^*(\uparrow j_q : p) = true$, this implies $I^*(\uparrow j' : p) = true$. □

The clauses E^*_{ijk} are tautological (and redundant) if $i \geq j$ or $j \geq i$; in particular, they are not needed if the ordering $>$ on the truth value set N is total.

According to the following lemma, it is not necessary to include in Γ^* clauses of the form $\uparrow i_1 : p, \ldots, \uparrow i_q : p \;\to\; \uparrow \bigsqcup\{i_1, \ldots, i_q\} : p$ for $q > 2$:

Lemma 2. *Let Γ be a regular Horn formula over a signature Σ, let I^* be a classical interpretation satisfying Γ^*, let $p \in \Sigma$, and let $M \subset N$ be a non-empty set of truth values such that $I^*(\uparrow j : p) = true$ for all $j \in M$; then $I^*(\uparrow \bigsqcup M : p) = true$.*

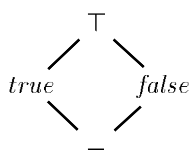
Proof. The lemma is proven by induction on the size of M.

$M = \{j\}$: Since j is the only element of M, we have $\bigsqcup M = j$; and because $I^*(\uparrow j : p) = true$ for all $j \in M$ this immediately implies $I^*(\uparrow \bigsqcup M : p) = true$.

$M \to M \cup \{j\}$: By assumption $I^*(\uparrow j : p) = true$. The induction hypothesis applies to M; thus, $I^*(\uparrow \bigsqcup M : p) = true$. If $j \geq \bigsqcup M$ then $\bigsqcup(M \cup \{j\}) = j$, and if $\bigsqcup M \geq j$ then $\bigsqcup(M \cup \{j\}) = \bigsqcup M$; in both cases, $I^*(\uparrow \bigsqcup(M \cup \{j\}) : p) = true$. Otherwise, if neither $j \geq \bigsqcup M$ nor $\bigsqcup M \geq j$, then Γ^* contains the clause $E^*_{j,j',k}$, where $j' = \bigsqcup M$ and $p = p_k$. As the interpretation I^* satisfies Γ^*, it satisfies in particular $E^*_{j,j',k}$. Thus, $I^*(\uparrow (j \sqcup \bigsqcup M) : p) = true$. As $j \sqcup \bigsqcup M = \bigsqcup(M \cup \{j\})$, this implies $I^*(\uparrow \bigsqcup(M \cup \{j\}) : p) = true$. □

Example 2. Assume that $N = \{-, \top, \mathit{true}, \mathit{false}\}$ and the partial order over N is the lattice shown on the right below. Given a regular Horn formula Γ (over signature Σ), for each propositional variable p occurring in Γ we add the following classical Horn clauses (over the signature Σ^*) to obtain Γ^*:

$$
\begin{array}{ll}
(D^*_1) & \uparrow \top : p \;\to\; \uparrow \mathit{true} : p \\
(D^*_2) & \uparrow \top : p \;\to\; \uparrow \mathit{false} : p \\
(D^*_3) & \uparrow \mathit{true} : p \;\to\; \uparrow - : p \\
(D^*_4) & \uparrow \mathit{false} : p \;\to\; \uparrow - : p \\
(E^*_1) & \uparrow \mathit{true} : p, \uparrow \mathit{false} : p \;\to\; \uparrow \top : p
\end{array}
$$



4.2 Results

The size of Γ^* is easily seen to have

$$|\Gamma| + 3s|N|^2$$

as an upper bound where s is the number of different atoms occurring in Γ.

As $s < |\Gamma|$, this places $|\Gamma^*|$ in $\mathcal{O}(|\Gamma||N|^2)$; and, since the time complexity of constructing Γ^* is linear in its size, the reduction * is in $\mathcal{O}(|\Gamma||N|^2)$.

If the ordering $\geq$ on the truth value set is total, the size of Γ^* is bounded by

$$|\Gamma| + 2s|N| \ ,$$

because in that case there are only $|N|$ many clauses D_{ijk} for each k in Γ^*, and no clauses E_{ijk} are needed. Then, * is in $\mathcal{O}(|\Gamma||N|)$.

Theorem 3. *The transformation * is computable in time linear in the size of the transformed formula and quadratic in the size of the truth value set.*

If the ordering $\geq$ is total, it is linear in both the size of the transformed formula and the size of the truth value set.

Because classical Horn SAT is linear [3], we get the following new result as a corollary:

Corollary 2. *Regular Horn SAT can be solved in time linear in the size of the formula and quadratic in the size of the truth value lattice.*

In the special case of totally ordered truth values, regular Horn SAT is of even smaller complexity (which was already known, see [8]).

Corollary 3. *Regular Horn SAT with totally ordered truth values can be solved in time linear in both the size of the formula and the size of the truth value set.*

4.3 Correctness

The following theorem states the correctness of the transformation *:

Theorem 4. *A (lattice-based) regular Horn formula Γ is satisfiable if and only if Γ^* is satisfiable.*

Proof. 1. *Only-if-part:* Assume that the interpretation I satisfies Γ. Define the classical interpretation I^* as follows. For all $i \in N$ and all atoms $p \in \Sigma$, if $I(p) \geq i$ (i.e., if $I \models \uparrow i : p$), let $I^*(\uparrow i : p) = \mathit{true}$, otherwise let $I^*(\uparrow i : p) = \mathit{false}$. Obviously, I^* is a well defined classical interpretation for the signature Σ^*; it remains to be shown that $I^* \models \Gamma^*$.

a. $I^* \models C_m$*:* By definition, the classical interpretation I^* satisfies an atom $\uparrow i{:}p \in \Sigma$ (regarded as an element of Σ^*) if and only if the interpretation I satisfies $\uparrow i{:}p$ (regarded as a signed atom). Thus, as I satisfies the clauses C_m, they are satisfied by I^* as well.

b. $I^* \models D^*_{ijk}$*:* Let $i, j \in N$ be arbitrary truth values such that $i > j$ and let p_k be an arbitrary atom occurring in Γ. Assume $I^*(\uparrow i : p_k) = \mathit{true}$; then $I(p_k) \geq i$ and, thus, $I(p_k) \geq j$. Therefore, $I^*(\uparrow j : p_k) = \mathit{true}$; and I^* satisfies the clause $D^*_{ijk} = \uparrow i : p_k \rightarrow \uparrow j : p_k$.

c. $I^* \models E^*_{ijk}$: Let $i, j \in N$ be arbitrary truth values and let p_k be an arbitrary atom occurring in Γ. Assume that both $I^*(\uparrow i : p_k) = \mathit{true}$ and $I^*(\uparrow j : p_k) = \mathit{true}$; then $I(p_k) \geq i$, $I(p_k) \geq j$ and, thus, $I(p_k) \geq i \sqcup j$. Thus, $I^*(\uparrow i \sqcup j : p_k) = \mathit{true}$, which implies that I^* satisfies the clause E^*_{ijk}.

2. *If-part:* Assume that the classical interpretation I^* satisfies Γ^*. Define the interpretation I for all propositional variables $p \in \Sigma$ by

$$I(p) = \bigsqcup \{i \in N \mid I^*(\uparrow i : p) = \mathit{true}\}$$

(by definition, $\bigsqcup \emptyset = -$). The interpretation I is well defined because in a lattice the supremum $\bigsqcup$ is well defined.

It remains to be shown that $I \models \Gamma$. For that is suffices to show that, for all truth values $i \in N \setminus \{-\}$ and all atoms p occurring in Γ: I satisfies $\uparrow i{:}p$ (regarded as a signed atom) if and only if I^* satisfies $\uparrow i : p$ (regarded as a classical atom in Σ^*); because then, if C is an arbitrary clause in Γ and I satisfies all atoms in the body of C (otherwise C is trivially satisfied by I), then I^* satisfies all atoms in the body of C and, since $I^* \models C$, it satisfies the head of C, which then implies that I satisfies the head of C and, therefore, the whole clause C.

The case $i = -$ can be excluded, because we have made the assumption that the sign $\uparrow -$ does not occur in Γ.

a. To prove that $I \models \uparrow i : p$ if $I^* \models \uparrow i : p$, assume the latter, i.e., $I^*(\uparrow i : p) = \mathit{true}$. Then, by definition of I, we have $I(p) \geq i$, which immediately implies that $I \models \uparrow i : p$.

b. To prove that $I \models \uparrow i : p$ only if $I^* \models \uparrow i : p$, assume the former, i.e., $I(p) \geq i$; by definition of I this implies $\bigsqcup M \geq i$ where $M = \{j \in N \mid I^*(\uparrow j : p) = \mathit{true}\}$. If M were empty, then $\bigsqcup M = -$ and, thus, $- \geq i$; this, however, would imply $i = -$, which contradicts the assumption $i \neq -$. Thus M is non-empty, and Lemma 2 applies to M; thus, we have $I^*(\uparrow \bigsqcup M : p) = \mathit{true}$. Now, since $\bigsqcup M \geq i$, the Lemma 1 applies to the truth values $\bigsqcup M$ and i. Therefore, $I^*(\uparrow i : p) = \mathit{true}$, i.e., $I^* \models \uparrow i : p$. □

4.4 Regular Unit Resolution

In this section we define a regular unit resolution calculus and prove its completeness for regular Horn clauses. The calculus is based on the following inference rules:

Positive Regular Unit Resolution (PRUR)

$$\frac{\begin{array}{l} \to \uparrow i : p \\ \uparrow i_1 : p_1, \ldots, \uparrow i_l : p, \ldots, \uparrow i_k : p_k \ \to \ \uparrow j : q \end{array}}{\uparrow i_1 : p_1, \ldots, \uparrow i_{l-1} : p_{l-1}, \uparrow i_{l+1} : p_{l+1}, \ldots, \uparrow i_k : p_k \ \to \ \uparrow j : q}$$

provided that $i \geq i_l$.

Regular Reduction (RR)

$$\frac{\begin{array}{l} \to \uparrow i : p \\ \to \uparrow j : p \end{array}}{\to \uparrow (i \sqcup j) : p}$$

provided that neither $i \geq j$ nor $j \geq i$.

Theorem 5. *A regular Horn formula Γ is unsatisfiable if and only if there exists a derivation of the empty clause from Γ using the calculus formed by the PRUR rule and the RR rule.*

Proof. Theorem 4 states that Γ is satisfiable if and only if Γ^* is satisfiable. Thus, we know that Γ is unsatisfiable if and only if there exists a derivation of the empty clause from Γ^* using classical positive unit resolution (PUR), since this rule is refutation complete for classical Horn formulas. We prove, by induction on the number n of deduction steps in that derivation using one of the additional clauses that are in Γ^* but not in Γ, that it is possible to construct from a classical (PUR) derivation of the empty clause from Γ^* a deduction of the empty clause from Γ using the PRUR and RR rule.

$n = 0$: In this case, the classical deduction using the PUR rule is a signed deduction using the PRUR rule.

$n - 1 \to n$: We concentrate on the last PUR deduction step involving one of the additional clauses in Γ^*; two situations can arise:

1. In that step $\rightarrow\, \uparrow j : p$ is deduced from $\rightarrow\, \uparrow i : p$ and $\uparrow i : p \rightarrow\, \uparrow j : p$, where $i \geq j$. This step is deleted from the derivation. To regain a proper deduction, in every PUR rule application with $\rightarrow\, \uparrow j : p$ as one of the premisses, this premiss is replaced by $\rightarrow\, \uparrow i : p$, which yields a proper PRUR application.

2. In that step $\uparrow j : p \rightarrow\, \uparrow (i \sqcup j) : p$ is deduced from the clauses $\rightarrow\, \uparrow i : p$ and $\uparrow i : p, \uparrow j : p \rightarrow\, \uparrow (i \sqcup j) : p$, where neither $i \geq j$ nor $j \geq i$. This clause is only relevant if it is used later on for deriving $\rightarrow\, \uparrow (i \sqcup j) : p$ from $\rightarrow\, \uparrow j : p$ and $\uparrow j : p \rightarrow\, \uparrow (i \sqcup j) : p$; otherwise we can delete this step. Both steps can be replaced by an application of the RR rule to the clauses $\rightarrow\, \uparrow i : p$ and $\rightarrow\, \uparrow j : p$.

In both cases at least one critical usage of one the additional clauses in Γ^* is removed, and the induction hypothesis applies to the resulting derivation. □

Our regular reduction rule can be seen as an improvement of the reduction rule presented in [9]. Whereas they provide a top down, Prolog-like proof procedure, we have defined a bottom-up procedure based on unit resolution. An alternate solution with an extended notion of signs that avoids reduction rules altogether can be found in [10]. In [9, 10], however, complexity issues are not discussed.

4.5 Infinite Truth Value Lattices

All the results of this section so far have only been proven for *finite* truth value lattices; for example, it is essential for the proof of Lemma 2 that the set of truth values is finite.

Nevertheless, the results apply in many cases to *infinite* truth value lattices as well, because it suffices to consider the sub-lattice that is generated by the truth values actually occurring in a formula and the bottom element.

Definition 7. *Given a regular Horn formula Γ over a (possibly infinite) truth value lattice $(N, \geq)$, let $(N_\star, \geq)$ be the sub-lattice of $(N, \geq)$ generated by the elements*

$$\{i \in N \mid i \text{ occurs in } \Gamma\} \cup \{-\}$$

The following theorem states that if the satisfiability of a formula Γ is to be checked, it suffices to only consider the truth value lattice $(N_\star, \geq)$. Thus, if $N_\star$ is finite and effectively computable for all Γ, then all results of this section can be made use of by considering the lattice $(N_\star, \geq)$ instead of $(N, \geq)$.

Theorem 6. *Let Γ be a regular Horn formula over a (possibly infinite) truth value lattice $(N, \geq)$. The formula Γ is satisfiable by an interpretation over the lattice $(N, \geq)$ if and only if it is satisfiable over the lattice $(N_\star, \geq)$.*

Proof. The if-part of the theorem is trivially true, because every interpretation over $(N_\star, \geq)$ is an interpretation over $(N, \geq)$ as well.

To prove the only-if part, assume that the N-interpretation I satisfies Γ. Define the $N_\star$-interpretation $I_\star$ for all atoms $p \in \Sigma$ by $I_\star(p) = \bigsqcup M_p$ where $M_p = \{i \in N_\star \mid I(p) \geq i\}$. It suffices to show that for all truth values i occurring in Γ (and thus in $N_\star$): $I \models \uparrow i : p$ if and only if $I_\star \models \uparrow i : p$.

a. Assume that $I \models \uparrow i : p$, i.e., $I(p) \geq i$. Then, $i \in M_p$ and, by definition of $I_\star$, we have $I_\star(p) \geq i$ and, thus, $I_\star \models \uparrow i : p$.

b. Assume that $I_\star \models \uparrow i : p$, i.e., $I_\star(p) \geq i$. Since $I(p) \geq j$ for all $j \in M_p$, we have $I(p) \geq \bigsqcup M_p = I_\star(p) \geq i$ and, thus, $I \models \uparrow i : p$ (note that the supremum operator $\sqcup$ is the same in both lattices). □

Since the formula Γ is finite, the set of elements generating $(N_\star, \geq)$ is finite as well. Therefore, the sub-lattice $(N_\star, \geq)$ is finite if $(N, \geq)$ is *locally finite*, i.e., if every sub-lattice generated by a finite subset is finite. This is, for instance, the case if the lattice $(N_\star, \geq)$ is distributive.

4.6 Extension to Partial Orders with Maximum

One of the main advantages of our transformational approach to signed logic is that it becomes completely transparent which additional deductive machinery is required as compared to the classical case. This becomes clearer even when we go beyond lattice-based regular Horn formulae.

We start with two considerations that somewhat limit the terrain. A core feature of any efficient deduction procedure for Horn formulae is the possibility to represent the conjunction of two unit clauses as a single unit clause as witnessed by the reduction rule in the previous section. This amounts to saying that signs of atoms must be closed under conjunction. When signs are upsets this condition can be expressed as:

$$\text{For all } i, j \in N \text{ there is a } k \in N \text{ such that } \uparrow i \cap \uparrow j = \uparrow k \tag{1}$$

It is easy to show that non-empty, finite posets with (1) are already upper semi-lattices. Therefore, it is inevitable to generalize the language of signs if we want to go beyond lattices.

Proof. Assume there was no $m \in N$ such that $m \geq i$ and $m \geq j$. Then $\uparrow i \cap \uparrow j = \emptyset$, contradiction.

Now assume i, j are covered by incomparable $m, m' \in N$ with $m \neq m'$ and $\uparrow i \cap \uparrow j = \uparrow k$. Then $\{m, m'\} \subseteq \uparrow k$. As m, m' are incomparable, k is different from both. On the other hand, by definition of k, $k \geq i$ and $k \geq j$, so m, m' do not cover i, j. The proof is illustrated in Fig. 1. □

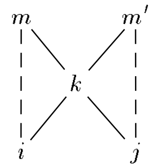


Fig. 1.

A natural candidate for an enriched language of signs are finite unions of upsets which can also be seen as finitely generated filters. In the following we write $\uparrow \{i_1, \dots, i_k\}$ instead of $\uparrow i_1 \cup \dots \cup \uparrow i_k$ and similar for $\downarrow$. We extend our notion of regularity (and hence of Horn formulae) as follows:

Definition 8. *If a sign S is of the form $\uparrow \{i_1, \dots, i_k\}$ or $\downarrow \{i_1, \dots, i_k\}$ for some $\{i_1, \dots, i_k\} \subseteq N$ and $k \geq 1$, then it is called a* regular sign*.*

A signed clause is called regular *if it contains regular atoms with signs only of the form $\uparrow \{i_1, \dots, i_k\}$.*[4] *A signed CNF formula is called regular if it only contains regular clauses. A regular clause containing at most one regular atom with positive polarity is a* regular Horn clause*. A regular CNF formula consisting solely of regular Horn clauses is a* regular Horn formula*.*

The next question is which partial orders can be captured if we want to retain an efficient decision procedure for regular Horn formulae. One such necessary condition is that there must be a maximum $\top$. To see this consider $N = \{\mathit{false}, \mathit{true}\}$ with $\mathit{false} \not\leq \mathit{true}$ and $\mathit{true} \not\leq \mathit{false}$. In this case $\{\mathit{false}\} = \uparrow \{\mathit{false}\}$ and $\{\mathit{true}\} = \uparrow \{\mathit{true}\}$, so each classical CNF clause can be expressed as a regular Horn clause. Given $C = p_1, \dots, p_k \to q_1, \dots, q_l$, simply rewrite C, for example, into:

$$\uparrow \{\mathit{true}\} : p_1, \dots, \uparrow \{\mathit{true}\} : p_k, \uparrow \{\mathit{false}\} : q_1, \dots, \uparrow \{\mathit{false}\} : q_{l-1} \to \uparrow \{\mathit{true}\} : q_l$$

Hence, we cannot expect to obtain a polynomial decision procedure for such Horn formulae. The problem is that by conjoining regular signs $\uparrow \mathit{false}$ and $\uparrow \mathit{true}$ we can express falsity at any time on the object level which is as good as to admit contrapositives of clauses.

Finally, we sketch how partial orders with maximum lead to a reasonable notion of generalized Horn formulae. This can be done via a reduction to lattice-based Horn formulae handled in the previous sections. For a partial order $(N, \leq)$ with maximum $\top$ consider the lattice $\mathcal{F}^+(N)$ of its non-empty order filters. Its elements can be represented as the non-empty anti-chains of $(N, \leq)$, that is

$$\{S \mid \emptyset \neq S \subseteq N, \text{ for all } i, j \in S: \text{ if } i \neq j \text{ then } i \not\leq j,\ j \not\leq i\} \ .$$

The order $\sqsubseteq$ on $\mathcal{F}^+(N)$ is defined as $S \sqsubseteq S'$ iff $\Uparrow S \supseteq \Uparrow S'$, where $\Uparrow S$ is the filter generated by S in N. To apply the results of the previous sections it is sufficient to show:

Proposition 1. *A regular literal $\uparrow S : p$ is satisfiable w.r.t. a poset with maximum $(N, \leq)$ iff it is satisfiable w.r.t. the lattice $\mathcal{F}^+(N)$.*[5]

[4] The remark at the end of Section 2.2 applies here as well.

[5] In the latter case, of course, S is interpreted as a single lattice element in $\mathcal{F}^+(N)$.

Proof. Assume $I \models \uparrow S : p$ in $(N, \leq)$, say $I(p) = i \in \uparrow S$. By a standard result on posets [2], $\Uparrow S = \Uparrow S'$ for an anti-chain S'. Every singleton over N is an anti-chain, so $\{i\}$ is an element of $\mathcal{F}^+(N)$. Moreover, $S' \sqsubseteq \{i\}$, therefore we have $I^* \models \uparrow S : p$ in $\mathcal{F}^+(N)$, with $I^*(p) = \{i\}$.

Vice versa, assume $I \models \uparrow S : p$ in $\mathcal{F}^+(N)$, with $I(p) = S' \in \uparrow S : p$, so $\Uparrow S' \subseteq \Uparrow S$ by definition. Let $i \in S'$ be arbitrary. We have $i \in \Uparrow S' \subseteq \Uparrow S$, hence with $I^*(p) = i$ we certainly have that $I^* \models \uparrow S : p$ in $(N, \leq)$. $\square$

There is a price to pay for the increased generality: The lattice $\mathcal{F}^+(N)$ can be considerably larger than the poset $(N, \leq)$, in the worst case exponentially larger. This proves:

Theorem 7. *Regular Horn SAT formulae based on posets with maximum can be solved in time linear in the size of the formula and exponential in the size of the truth value set.*

Example 3. Consider the poset N depicted on the left in Fig. 2. The lattice $\mathcal{F}^+(N)$ is shown on the right. It can be seen as a lattice-completion of N.

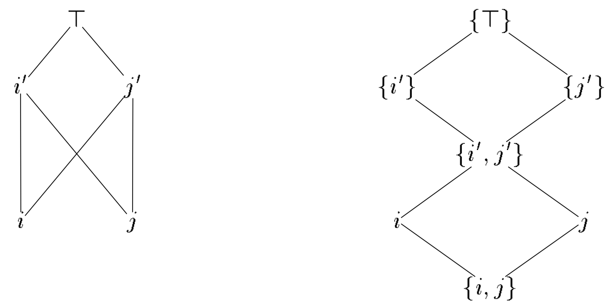


Fig. 2.

From a deductive point of view it is important to compute the supremum $\sqcup$ and $\sqsubseteq$ in $\mathcal{F}^+(N)$, because these are required in the reduction and unit resolution rule, respectively.

Let $\Uparrow S = \Uparrow \{i_1, \ldots, i_k\}$ and $\Uparrow S' = \Uparrow \{j_1, \ldots, j_l\}$ be given. We denote with $max(i, j)$ the set of minimal elements above i and j in N w.r.t. $\leq$. Now $S \sqcup S' = \Uparrow S \cap \Uparrow S' = \{k \mid k \in \max(i, j),\ i \in S,\ j \in S'\}$. From the resulting set any elements not minimal in it can be deleted to arrive at an anti-chain representation. Finally, $S \sqsubseteq S'$ iff $\Uparrow S \supseteq \Uparrow S'$ iff for all $j_r \in S'$ there is a $i_s \in S$ such that $i_s \leq j_r$.

5 Future Work

An investigation of the lattice theoretic aspects of lattice-based regular Horn formulae could lead to useful new results. In particular, the infinite case, for which only first ideas have been presented in Section 4.5, representation theory and dualities should be further studied.

As another line of work, experiments should be carried out to compare specific decision procedures for regular Horn formulae with procedures based on applying the transformation * defined in Section 4 and then using a procedure for classical Horn formulae.

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